NASA TN D-227

NASA TN D-227

7N-13

198493

40P.

TECHNICAL NOTE
D-227

EFFECT OF ECCENTRICITY OF THE LUNAR ORBIT, OBLATENESS OF THE EARTH, AND SOLAR GRAVITATIONAL FIELD ON LUNAR TRAJECTORIES

By William H. Michael, Jr., and Robert H. Tolson

Langley Research Center
Langley Field, Va.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

WASHINGTON June 1960

1D

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

TECHNICAL NOTE D-227

EFFECT OF ECCENTRICITY OF THE LUNAR ORBIT, OBLATENESS OF THE EARTH, AND SOLAR GRAVITATIONAL FIELD ON LUNAR TRAJECTORIES

By William H. Michael, Jr., and Robert H. Tolson

L
8
0
0

SUMMARY

Calculations have been made to determine the magnitude of the effects on lunar trajectories of the eccentricity of the lunar orbit, the oblateness of the earth, and the solar perturbation. Comparisons have been made between lunar trajectories which were calculated by using the classic restricted three-body equations of motion and lunar trajectories (with identical injection conditions) which were calculated by using equations of motion which include terms representing the additional effects. On the basis of such comparisons, it was found that the oblateness of the earth can modify the trajectories in the vicinity of the moon by several hundred miles or more; whereas, the eccentricity of the moon's orbit and the gravitational gradient of the sun are relatively less important and cause a difference in impact point of no more than one or two hundred miles along the lunar surface. The results of such comparisons give an indication of whether these additional considerations should be included in trajectory calculations for particular lunar missions.

INTRODUCTION

With the rapid development of space technology, prospects of lunar exploration have advanced from the state of remote possibility to that of fairly imminent probability. In parallel with the advance in technology, analytical studies of lunar trajectories have become more sophisticated. Early lunar-trajectory studies, including those of references 1 to 4, considered the motion of an infinitesimal body under the gravitational attractions of the point masses of the earth and moon. In these studies, the earth and moon were considered to rotate in circles about the center of mass of the earth-moon system, and the motion of the small body was confined to the plane defined by the motion of the earth and moon. Such two-dimensional studies have been followed more recently by three-dimensional trajectory studies which eliminate the restriction of confining the motion of the space vehicle to the earth-moon plane. The three-dimensional studies, some of which are discussed in references 5 to 7, allow consideration of such parameters as latitude of the injection point, injection azimuth angle, declination of the moon, and inclination of the plane of the vehicle trajectory to the earth-moon plane.

The two- and three-dimensional studies discussed above for the most part do not include the effects on the trajectories of the eccentricity of the moon's orbit, the oblateness of the earth, or the gravitational
field of the sun. In reference 1, however, is included a discussion of L
the approximate effects of such additional considerations, and refer- 8
ence 2 includes some trajectory calculations which investigate the effect 0
of the solar perturbation on a circumlunar mission. For particular lunar 0
missions in which precise trajectory calculations are required, these effects should be included. On the other hand, industrial and research organizations have no doubt accumulated a large number of trajectory calculations in which these effects have been neglected, and it is of interest to determine to what extent inclusion of the additional effects might influence the results previously obtained. If the effects of inclusion of the lunar orbit eccentricity, earth's oblateness, and solar perturbation are small, it is possible that such effects can be neglected for a variety of lunar missions, and results of previous calculations or the simplified analyses can be used.

The present paper compares results of lunar trajectories which include and neglect the effects of the eccentricity of the moon's orbit, the earth's oblateness, and the solar perturbation to determine the order of magnitude of these effects on lunar trajectories. The comparisons are made for lunar trajectories which impact the moon, and differences in impact points on the lunar surface due to the additional effects are given. The results are also applicable as indications of the order of magnitude of the perturbation effects on lunar trajectories for other lunar missions.

SYMBOLS

a	distance between centers of mass of earth and moon, miles
B_x,B_y,B_z	nondimensional gravitational perturbation acceleration components due to earth's oblateness
F_x,F_y,F_z	perturbation acceleration components due to sun's potential gradient, miles/hr^2
i_m	inclination of equatorial plane to earth-moon plane, deg

i_s inclination of ecliptic to earth-moon plane, deg

J coefficient of second harmonic of earth gravitational potential, 0.001624

$$k_1 = \frac{\text{Mass of earth}}{\text{Mass of earth + Mass of moon}} = \frac{81.45}{82.45}$$

$$k_2 = -\frac{\text{Mass of moon}}{\text{Mass of earth + Mass of moon}} = -\frac{1}{82.45}$$

L
3
0
0

R equatorial radius of earth, 3963.34 miles

R_s radius from origin of coordinates to sun, 92.9×10^6 miles

r_k distance from vehicle to kth body, miles

t time from injection, hr

u_m angular position of the moon measured eastward from X-axis, deg

u_s angular position of sun in ecliptic measured eastward from ascending node, deg

V injection velocity, mph

V_p parabolic velocity at injection altitude, 24,125.8 mph at 300 miles altitude

x,y,z position components of vehicle

x_k,y_k,z_k position components of kth body

x_m,y_m,z_m position components of vehicle relative to moon (see fig. 1)

θ geocentric colatitude of vehicle, deg

ϕ gravitational potential of earth, $\frac{\text{miles}^2}{\text{hr}^2}$

μ_k product of universal gravitational constant and mass of kth body, $\frac{\text{miles}^3}{\text{hr}^2}$

Ω_m angle measured in XY plane eastward from X-axis to node of equator, deg

Ω_s angle measured in XY plane eastward from X-axis to ascending node of sun's plane, deg

γ vehicle injection angle, angle between velocity vector and normal to radius vector, deg

ψ vehicle position angle, angle between geocentric radius vector at injection and X-axis, measured in earth-moon plane, deg

ρ initial angle between vehicle plane and earth-moon plane, deg

L
8
0
0

Subscripts:

1 earth

2 moon

3 sun

EQUATIONS OF MOTION

The equations of motion of an infinitesimal body moving under the gravitational attraction of the earth and moon have been modified to include the additional effects of the eccentricity of the moon's orbit, the oblateness of the earth, and the gravitational attraction of the sun. The equations are written with respect to a coordinate system with the X-axis as the line from the center of the earth to the center of the moon at the time of injection, the Y-axis normal to the X-axis in the plane of motion of the moon and positive in the direction of motion, and the Z-axis normal to the plane of motion of the moon. The coordinate system, which is shown in figure 1, has its origin at the center of mass of the earth-moon system.

The equations of motion of the vehicle are:

$$\left.\begin{aligned}
\frac{d^2x}{dt^2} &= -\mu_1 \frac{(x - x_1)(1 + B_x)}{r_1^{\,3}} - \mu_2 \frac{(x - x_2)}{r_2^{\,3}} + F_x \\
\frac{d^2y}{dt^2} &= -\mu_1 \frac{(y - y_1)(1 + B_y)}{r_1^{\,3}} - \mu_2 \frac{(y - y_2)}{r_2^{\,3}} + F_y \\
\frac{d^2z}{dt^2} &= -\mu_1 \frac{z(1 + B_z)}{r_1^{\,3}} - \mu_2 \frac{z}{r_2^{\,3}} + F_z
\end{aligned}\right\} \tag{1}$$

L 800

where

$$r_1 = \sqrt{(x - x_1)^2 + (y - y_1)^2 + z^2}$$

and

$$r_2 = \sqrt{(x - x_2)^2 + (y - y_2)^2 + z^2}$$

The product of the mass of the earth and the universal gravitational constant μ_1 is taken as $1.239483 \times 10^{12} \frac{\text{miles}^3}{\text{hr}^2}$. The earth-to-moon mass ratio is taken as 81.45. The B and F terms in the equations represent the perturbation accelerations on the vehicle due to the earth's oblateness and the gravitational field of the sun, respectively.

The coordinates of the earth $(x_1,y_1,0)$ and the moon $(x_2,y_2,0)$ are time dependent. The equations for these coordinates are:

$$x_1 = k_2 a \cos u_m \qquad\qquad y_1 = k_2 a \sin u_m$$

$$x_2 = k_1 a \cos u_m \qquad\qquad y_2 = k_1 a \sin u_m$$

where a and u_m are tabulated as functions of time. For circular lunar orbits, a is constant and u_m is a linear function of time. For eccentric lunar orbits, these parameters are calculated from data given in reference 8. Thus, comparisons of lunar trajectories can be made in which the orbit of the moon is considered to be either circular or noncircular.

For consideration of the oblateness of the earth, the earth is taken as an oblate spheroid. It is further assumed that the first two terms of the expansion of the potential will give sufficient accuracy for this investigation. With these assumptions, the earth gravitational potential function is

$$\phi = -\frac{\mu_1}{r_1}\left[1 + \frac{JR^2}{r_1^2}\left(\frac{1}{3} - \cos\theta\right)\right]$$

where the first term in the bracket is the potential function for the spherical earth and the following terms represent the oblateness effect. The gravitational force components along the coordinate axes are the negative gradients of the potential; for instance, the force along the X-axis is given by

$$-\frac{\partial\phi}{\partial x} = -\frac{\partial\phi}{\partial r_1}\frac{\partial r_1}{\partial x} - \frac{\partial\phi}{\partial\theta}\frac{\partial\theta}{\partial x}$$

The geocentric colatitude of the vehicle can be obtained from the position coordinates of the vehicle and the geometry of the situation. The expression for the geocentric colatitude of the vehicle is

$$\cos\theta = -\frac{1}{r_1}\left[-\left(x - x_1\right)\sin i_m \sin\Omega_m + \left(y - y_1\right)\sin i_m \cos\Omega_m - z\cos i_m\right]$$

The angular relations are shown in figure 2. After performing the indicated differentiation and combining terms, the nondimensional perturbation acceleration components due to the earth's oblateness are

$$B_x = \frac{JR^2}{r_1}\left(\frac{1 - 5\cos^2\theta}{r_1} + \frac{2\cos\theta\sin i_m \sin\Omega_m}{x - x_1}\right)$$

$$B_y = \frac{JR^2}{r_1}\left(\frac{1 - 5\cos^2\theta}{r_1} - \frac{2\cos\theta\sin i_m\cos\Omega_m}{y - y_1}\right)$$

$$B_z = \frac{JR^2}{r_1}\left(\frac{1 - 5\cos^2\theta}{r_1} + \frac{2\cos\theta\cos i_m}{z}\right)$$

The value for J is taken as 0.001624 from reference 9.

The solar acceleration of the vehicle with respect to the inertial coordinate system is the vector difference between the solar acceleration of the vehicle and the solar acceleration of the origin of the coordinate system. The components of the perturbation accelerations have been determined by standard perturbation methods from considerations similar to those discussed in reference 10. The terms to be entered into equation (1) to account for the solar gravitational field are

$$F_x = -\mu_3\left[\frac{x}{r_3^3} - x_3\left(\frac{1}{r_3^3} - \frac{1}{R_s^3}\right)\right]$$

$$F_y = -\mu_3\left[\frac{y}{r_3^3} - y_3\left(\frac{1}{r_3^3} - \frac{1}{R_s^3}\right)\right]$$

$$F_z = -\mu_3\left[\frac{z}{r_3^3} - z_3\left(\frac{1}{r_3^3} - \frac{1}{R_s^3}\right)\right]$$

where

$$r_3 = \sqrt{(x - x_3)^2 + (y - y_3)^2 + (z - z_3)^2}$$

The product of the sun's mass and the universal gravitational constant μ_3 is taken as $4.15 \times 10^{17} \frac{\text{miles}^3}{\text{hr}^2}$. The sun is assumed to revolve about the origin with constant angular velocity, in a plane inclined at a constant angle to the earth-moon plane and at a constant radius from the origin. The equations used to specify the coordinates of the center of mass of the sun are

$$x_3 = R_s\left(\cos u_s \cos \Omega_s - \sin u_s \sin \Omega_s \cos i_s\right)$$

$$y_3 = R_s\left(\cos u_s \sin \Omega_s + \sin u_s \cos \Omega_s \cos i_s\right)$$

$$z_3 = R_s \sin u_s \sin i_s$$

where u_s is the angular position of the sun in the ecliptic and increases at a rate of about 0.041 deg/hr. The value of u_s is computed as a phase angle plus the product of the angular rate and the time from injection. (See fig. 3 for angular descriptions.)

The equations of motion, equations (1), along with the auxiliary equations have been programmed for numerical integration by a fourth-order Runge-Kutta procedure on a digital computing machine. The results of the integration process are the position and velocity components of the vehicle as functions of time. Additional results include the position components of the vehicle relative to a rotating coordinate system $\left(X_m, Y_m, Z_m\right)$ fixed at the center of the moon as shown in figure 1.

The equations can be reduced to the classical, three-body equations by neglecting the terms due to the earth's oblateness and the solar perturbation and by entering as constants the earth-moon distance and rate of revolution of the moon in its orbit. Thus, these equations provide a means for comparison of trajectory calculations with and without the effects of the eccentricity of the moon's orbit, the oblateness of the earth, and the solar gravitational field.

BASIC CASES FOR TRAJECTORY COMPARISONS

For determining the order of magnitude of the effects of the eccentricity of the moon's orbit, the oblateness of the earth, and the solar gravitational field on lunar trajectories, comparisons are made between trajectory calculations which include these effects and calculations in which such effects have been neglected. The lunar trajectories in which these effects are neglected are referred to as basic cases, and correspond to the classic restricted three-body problem of celestial mechanics in which an infinitesimal body moves under the gravitational influence of the point masses of two finite bodies which revolve in circles about their common center of mass at a constant angular rate.

Some typical lunar trajectories have been chosen for the basic cases. The basic cases have the common characteristics of injection from the latitude of 28.46° N. (Cape Canaveral, Fla.), and injection at an altitude of 300 statute miles. The injection conditions for the basic cases have been chosen to give lunar impact at specified design values of declination of the moon and to give the same initial inclination of the plane of the trajectory to the earth-moon plane for each set of velocity ratios. The basic cases cover a range of injection velocity for two values of earth-moon distances, one corresponding to perigee (229,100 miles) for a particular month and one corresponding to mean earth-moon distance (238,857 miles). For these basic cases, the rate of angular motion of the earth and moon about the center of mass of the earth-moon system was considered constant for a given earth-moon distance and was obtained from the Keplerian period of rotation of the two bodies at that distance. The injection conditions for the basic cases are given in table I. Typical trajectories are plotted with respect to the inertial coordinate system in figure 4.

For the investigation of the effect of the eccentricity of the moon's orbit, two sets of basic cases permit comparisons of the effect of the eccentricity at the earth-moon distances at which the rate of change of distance is a minimum (at perigee) and a maximum (at mean distance).

For the investigation of the perturbations due to the sun's gravitational field and the oblateness of the earth on lunar trajectories, the particular earth-moon distance is of relatively minor importance and therefore trajectory comparisons have been made for only one value (238,857 miles) of this distance. For the effect of the earth's oblateness, the geocentric latitude history of the vehicle is of importance, and some additional basic cases are considered, as discussed in a later section.

It should be pointed out that the basic cases have not been related to any particular date of firing. Although the launch latitude, declination of the moon at contact, and earth-moon distances at contact have been specified, these values have been chosen as fairly typical ones for lunar trajectories and are not necessarily applicable for any particular date. This also applies to the variation of the earth-moon distance used in the calculations for the eccentricity of the moon's orbit. Although the earth-moon distances for a particular month have been used as a typical case, the other parameters for the trajectories do not necessarily correspond to particular dates for that month.

EFFECTS OF THE ECCENTRICITY OF THE LUNAR ORBIT ON LUNAR TRAJECTORIES

General Considerations

Suppose that injection conditions relative to the earth have been determined for lunar trajectories which result in impact on the moon, assuming that the moon moves in a circular orbit about the earth. With these same injection conditions, it is desired to determine to what extent consideration of the eccentricity of the moon's orbit affects the lunar trajectories. (It is assumed throughout this discussion that the earth-moon distance at the design time of vehicle contact is the same for both the circular and eccentric lunar orbits.)

L
8
0
0

The fact that the moon moves in an eccentric orbit means that its angular rate of motion about the earth and the distance between the earth and moon are changing with time. It is thus apparent that during the flight of the vehicle from earth to moon the positions of the moon in its true eccentric orbit differ somewhat from the positions which it would have in a circular orbit. These differences in the moon's position introduce a perturbation which can be expected to cause changes in the vehicle trajectories.

To investigate the effects of the eccentricity of the moon's orbit on lunar trajectories, trajectory calculations which neglect and include the eccentricity of the orbit are compared. (For these comparisons, the earth's oblateness and solar perturbation terms in eqs. (1) are neglected in both the eccentric and circular orbit trajectory calculations.) For the eccentric orbit trajectory calculations, the actual earth to moon distances and the angular position of the moon in the earth-moon plane are entered into a table as functions of time. The values for these parameters are calculated from data available in reference 8. The earth-moon distance is calculated from the horizontal parallax of the moon. The angular motion of the moon in the earth-moon plane involves reduction of the data for latitude and longitude of the moon (in the ecliptic system) to angular position in the earth-moon plane.

For the month of September 1959, the earth-moon distance is shown at the top of figure 5, and the angular rate of motion of the moon in the earth-moon plane is shown at the bottom of the figure. The earth-moon distances and angular rate of motion of the moon shown by the solid curves of figure 5 have been used as typical values in the trajectory calculations which consider the eccentricity of the moon's orbit. Also shown in the bottom part of the figure are the rates of angular motion of the moon assuming a circular orbit with radius equal to the earth-moon distance for the given date, such as might be used in the corresponding three-body problem. (The angular rates, for earth to moon distances of 238,857 miles and 229,100 miles, which are used in the basic cases are indicated on the figure.) The choice of perigee distance (229,100 miles) and mean earth-moon distance (238,857 miles) as the distances at time of lunar impact for the trajectory calculations considers the cases for which approximately minimum and maximum rate of change of earth-moon distance and minimum and maximum rate of change of angular rate of motion of the moon occur during the flight time to the moon.

Injection Situation for Eccentric-Orbit Trajectories

For comparing trajectories calculated with circular and eccentric lunar orbits, the calculations were made with equal values of injection conditions relative to the center of the earth. With regard to the position of the moon at injection, two different injection situations are considered for the eccentric-orbit calculations. These injection situations are illustrated schematically in figure 6.

One injection situation assumes that the position of the moon at injection, relative to the design contact position, is the same for the eccentric-orbit trajectories as for the comparable circular-orbit trajectories. This also means that the position angle ψ for the eccentric- and circular-orbit trajectories is the same. Trajectories calculated with this injection situation are called trajectories A and are illustrated schematically in part (a) of figure 6.

The other injection situation assumes that for both eccentric- and circular-orbit cases the moon is at the design contact position after the design flight time. The position angle ψ for the eccentric-orbit trajectory is thus different from that for the comparable circular-orbit case. However, for this situation the injection time, measured relative to the design contact time, is the same for both the eccentric-orbit case and the comparable circular-orbit case. Trajectories calculated with this injection situation are called trajectories B and are illustrated in part (b) of figure 6.

Trajectory Comparisons

The comparisons of lunar trajectories with circular and eccentric lunar orbits are shown in figures 7 to 9. In order to facilitate the presentation, these trajectories are plotted with respect to a right-hand rotating coordinate system with the origin at the center of the moon (fig. 1). In this right-hand system, the X_m-axis is the line joining the centers of the earth and moon, positive in the direction away from the earth; the Y_m-axis is normal to the X_m-axis, positive in the direction of the moon's motion; and the Z_m-axis is normal to the earth-moon plane. Two views of the trajectories are shown in figures 7 to 9, one looking down on the earth-moon plane and one looking along the earth-moon plane. Time of flight, measured in hours from the injection time, is indicated on the figures.

It is to be noted that all the trajectories shown in figures 7 to 9 result in lunar impact. This fact gives an indication of the order of magnitude of the effect of the eccentricity of the lunar orbit on lunar trajectories, for even in the cases considered in trajectories A, the effect of the eccentricity is not sufficiently large to prevent lunar impact.

On comparing trajectories A with the basic cases, it is seen that all these trajectories tend to impact at points more behind the center of the moon than the basic cases, partially because of the greater angular travel of the moon during the time of flight of the vehicle for the eccentric cases, as indicated by the angular rates of motion shown in figure 5. Additional considerations of such items as the gravitational influence of the moon on the trajectories, the variation of this gravitational influence with injection velocity and earth-moon distance, differences in flight time, and other factors tend to complicate any more detailed discussion of the trajectories.

Trajectories B are also presented in figures 7 to 9. Comparison of trajectories B with the basic trajectories shows that when the difference in angular motion of the moon during the flight time for the two cases is accounted for, the effect of the eccentricity of the lunar orbit is fairly small. The difference in angular motion is accounted for simply by injecting the vehicle, for the eccentric-orbit case, at the same time relative to the design contact time as that for the basic case. The differences between the basic trajectories and trajectories B are partially due to small differences in the gravitational force, in magnitude and direction, of the moon on the vehicle, and partially due to small differences in flight time. No attempt is made to evaluate the separate effects. The differences between impact points on the lunar surface for the basic trajectories and trajectories B amount to about 12 to 230 miles. The differences between impact points for trajectories A and B and the basic cases are summarized in table II.

From the preceding results, it can be seen that the effect of the eccentricity of the lunar orbit can be such as to modify lunar trajectories in the vicinity of the moon by an order of magnitude of about a lunar radius as compared with trajectories which consider a circular lunar orbit. However, if the injection situation for the eccentric orbit is chosen such that the injection time is the same, relative to the design contact time, as that for the basic case, the effect of the eccentricity of the orbit is small.

THE EFFECT OF THE EARTH'S OBLATENESS ON LUNAR TRAJECTORIES

General Considerations

To investigate the effect of the earth's oblateness, trajectories calculated by utilizing the equations of motion which include the additional terms due to the oblateness are compared with those trajectories calculated by neglecting these terms; that is, the basic cases. For both of these sets of trajectory calculations, the solar gravitational field and the eccentricity of the moon's orbit are neglected. The injection conditions for the trajectory calculations which include and neglect the oblateness are identical, and comparison of the two sets of trajectories in the vicinity of the moon gives an indication of the magnitude of the effect of the earth's oblateness.

The manner in which the oblateness perturbation affects the trajectory depends on a number of variables. The most important variables appear to be the heading angle and the latitude of the launch point which determine the inclination of the orbit to the equator, the injection velocity which essentially determines the flight time, and the injection angle which determines the geocentric angular travel of the vehicle. A rough qualitative description of the effects can be made by resolving the perturbation force into its two components and examining the effect on the trajectory of each component. The additional force components due to the earth's oblateness are:

(1) A radial force which produces a maximum acceleration toward the center of the earth when the particle is on the equator, produces a maximum acceleration away from the earth when the particle is at a pole, and vanishes at a latitude of 35.27°.

(2) An orthogonal force component (normal to the geocentric radius and in the plane containing the radius and the polar axis) which always acts to accelerate the vehicle toward the equatorial plane, has a maximum value at a latitude of 45°, and vanishes when the vehicle is at the pole or in the equatorial plane.

The force components are proportional to the coefficient of the second harmonic of the gravitational potential and inversely proportional to the fourth power of the geocentric radius of the particle. Thus, they are relatively small forces which decrease very rapidly as the particle recedes from the earth. Because of this, the forces produce their major effects just after the vehicle is injected and is still relatively near the center of the earth.

Trajectory Comparisons

Figure 10 shows the basic trajectories (1, 2, and 3 in table I) and the perturbed trajectories. The trajectories are terminated at the point at which they intercept the lunar surface. The distance measured along the lunar surface from the basic-trajectory impact point to the perturbed-trajectory impact point is tabulated in table II for the trajectories presented in figure 10. (Trajectories for the solar gravitational gradient perturbation are also shown in figure 10. These trajectories are discussed in a later section.)

L 8 0 0

For the cases presented in figure 10, which have a due-east injection, the trace of the vehicle on the surface of the earth rapidly approaches the equator during the important part of the trajectory. Figure 11 shows the latitude and geocentric-radius time histories for the basic case with a velocity ratio of 0.996 (basic case 2 in table I). The other velocity ratios have similar variations. During the first hours the radial force component tends to increase the curvature of the trajectory and to decrease the velocity as compared to the basic cases. The increased curvature displaces the trajectory in front of and below the moon and the reduced velocity displaces the trajectory behind the moon. Since the initial part of the trajectory is north of the equator, the orthogonal component tends to displace the trajectory below the moon. However, the trace of the vehicle rapidly approaches the equator at which the orthogonal component vanishes; therefore, the effect of this component is small.

The relative effectiveness of these forces will depend on a number of variables. For example, it is expected that a given retarding force will not change the flight time to the moon for high injection velocities as much as it will for low injection velocities. Also, the radial force will change the curvature of a low-injection-angle trajectory more than it will for a high-injection-angle trajectory inasmuch as the vehicle has a greater geocentric angular travel in the effective range of the force in the former case. In the light of the above discussion, it appears that for the basic cases presented in which the injection angle decreases as the velocity increases there may be an injection velocity at which the displacements of the trajectory in the XY plane will vanish. Figure 10 shows that the perturbed trajectory for $\frac{V}{V_p} = 1.006$ passes in front of the respective basic trajectory while the perturbed trajectory for $\frac{V}{V_p} = 0.996$ passes behind the basic trajectory; thus the velocity at which the displacements cancel is between these two values.

It is expected that the manner in which the oblateness perturbation disturbs the trajectory will depend to a certain extent on the latitude time history of the vehicle immediately after injection. In order to indicate the magnitude of this effect, three additional basic trajectories (cases 7, 8, and 9 in table I) were calculated with the use of the same velocity ratios and injection angles as cases 1, 2, and 3 in table I. However, the vehicle is injected with a heading north of east in order to impact the moon at zero declination. Figure 11 shows the latitude time history for $\frac{V}{V_p} = 0.996$ (basic case 8 in table I). The radius time histories for cases 2 and 8 are essentially the same.

L800

Figure 12 shows these basic trajectories (cases 7, 8, and 9 in table I) and also the perturbed trajectories in the region of the moon, and table II gives the distance measured on the lunar surface between the basic and perturbed impact points. In the region of the moon, the perturbed trajectories in figure 12 are displaced slightly more in the negative z direction than the perturbed trajectories presented in figure 10. This displacement is partly due to the increased magnitude of the orthogonal component as the initial parts of the trajectories pass farther north with respect to the equator. The perturbed trajectories in figure 12 also are displaced slightly farther behind the moon than those given in figure 10. This slightly greater displacement results from the additional effect that the orthogonal component has on the trajectory because the path is north of east at injection. This force now has components which both tend to decrease the velocity of the vehicle and to decrease the curvature of the trajectory, causing the vehicle to pass farther behind the moon.

The result of these additional displacements, as compared with the due-east injection cases, is to increase the impact-point variation by a few miles, as indicated in the numerical results in table II.

The results discussed in the preceding analysis indicate that the effect of the earth's oblateness on lunar trajectories can be of some importance, especially for trajectories with low injection velocities and latitude time histories which produce additive displacements in the region of the moon. As the injection velocity increases, the effect of the earth's oblateness decreases but is still of sufficient magnitude to be considered in a variety of lunar missions.

THE EFFECT OF THE SOLAR PERTURBATION ON LUNAR TRAJECTORIES

The effect of the solar perturbation is investigated by comparison of lunar trajectories calculated with use of equations of motion which include and neglect the terms due to the solar gravitational field.

The two sets of trajectories were calculated with identical injection conditions (with $a = 238{,}857$ miles), and for both sets the terms in the equations of motion due to the eccentricity of the moon's orbit and the oblateness of the earth are neglected. For the results presented herein, the sun was 5.2° North of the earth-moon plane, at a distance of 92.9×10^{6} miles from the barycenter, and was assumed to rotate about the barycenter with constant angular velocity and radius.

Various angular positions of the sun at injection were investigated to determine the location which gave maximum displacement of the impact point from the basic-case impact point. Figure 13 shows the results of this investigation for the median-velocity case $\left(\frac{V}{V_p} = 0.996\right)$. The angular coordinate gives the position of the sun projected into the earth-moon plane. The angle is measured eastward from the X-axis to the sun's position at injection. The radial coordinate indicates the displacement in miles on the lunar surface of the perturbed impact point from the basic-case impact point for a given position of the sun. For example, the maximum displacement on the lunar surface of 120 miles occurs when the sun is 50° east of the X-axis at injection. For this velocity ratio, the moon and sun rotate around the barycenter about 33° and 2.4°, respectively, during the flight time of 58 hours.

L
8
0
0

Let the solar perturbation force be resolved into tangential and normal components at each point along the trajectory. The tangential component accelerates the vehicle along the path, either increasing or decreasing the flight time and causing the vehicle to pass behind or in front of the center of the moon. The normal component essentially changes the curvature of the path. If the curvature is increased, the vehicle passes in front of the moon and if decreased the vehicle passes behind the moon. As the angular position of the sun is varied, these two effects change in relative importance and result in the maximums and minimums shown in figure 13. The maximums result from an additive combination of these forces and the minimums result from a combination in which the displacements due to each component tend to cancel.

The trajectories in the region of the moon for sun positions for which the maximum displacements occur are shown in figure 10. The arrows indicate the direction of the sun when the vehicle impacts on the lunar surface. Table II gives the magnitude of the impact-point variation for these three cases. The displacement of the perturbed trajectory from the basic trajectory is seen to decrease rapidly as the velocity increases or the flight time decreases.

In order to determine the effect on the impact point of the sun's inclination with respect to the XY plane, the inclination was varied from +5.2° to -5.2°. This variation produced only small changes in the direction and magnitude of the displacements on the lunar surface.

COMBINED EFFECTS OF PERTURBATIONS

Some additional calculations have been made to investigate the combined effects of the eccentricity of the lunar orbit, the oblateness of the earth, and the solar gravitational gradient on lunar trajectories. The results of the calculations indicate that the combined effects approximately correspond to a superposition, with respect to the basic cases, of the individual effects. Thus, if more than one of the individual effects are considered to be of importance for a particular mission, all the important effects should be included in the trajectory calculations.

CONCLUDING REMARKS

Comparisons have been made of lunar trajectories which have been calculated using the classic restricted three-body equations of motion and lunar trajectories (with identical injection conditions) which have been calculated with equations of motion which include terms representing the additional effects of the eccentricity of the moon's orbit, the oblateness of the earth, and the solar gravitational gradient.

The effect of the eccentricity of the lunar orbit can be such as to modify lunar trajectories in the vicinity of the moon by an order of magnitude of about a lunar radius as compared with trajectories which consider a circular orbit. However, if the injection situation for the eccentric orbit is chosen such that the injection time is the same, relative to the design contact time as that for the basic case, the effect of the eccentricity is small, of the order of one or two hundred miles on the lunar surface.

The effect of the earth's oblateness on lunar trajectories is dependent on the time history of the geocentric latitude of the vehicle, particularly during the time in which the vehicle is in the vicinity of the earth. The effect of the earth's oblateness can cause differences in the lunar trajectories with respect to the moon of several hundred miles. The effect of the earth's oblateness decreases as the injection velocity is increased.

The effect of the solar gravitational gradient on lunar trajectories is relatively small and causes differences in the trajectories with respect to the moon on the order of magnitude of one or two hundred miles in impact point on the lunar surface. These differences also decrease as the injection velocity is increased.

Whether consideration should be given to differences of the amounts discussed above depends on the particular lunar mission, but the present results indicate the order of magnitude of the differences in the trajectories due to the approximation involved in neglecting the additional effects.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Field, Va., November 17, 1959.

REFERENCES

1. Buchheim, R. W.: Motion of a Small Body in Earth-Moon Space. U.S. Air Force Project RAND Res. Memo. RM-1726 (ASTIA Doc. No. AD 123557), The RAND Corp., June 4, 1956.

2. Ehricke, Krafft A.: Cislunar Operations. [Preprint] 467-57, presented at the Am. Rocket Soc. Meeting (San Francisco, Calif.), June 10-13, 1957.

L
8
0
0

3. Lieske, H. A.: Accuracy Requirements for Trajectories in the Earth-Moon System. P-1022, The RAND Corp., Feb. 19, 1957.

4. Yegorov, V. A.: Certain Problems on the Dynamics of Flight to the Moon. Rep. No. 187 (Project NY-1377), U.S. Joint Pub. Res. Service, Jan. 25, 1958, pp. 101-174.

5, Michael, William H., Jr., and Tolson, Robert H.: Three-Dimensional Lunar Mission Studies. NASA MEMO 6-29-59L, 1959.

6. Goldbaum, G. C., and Gunkel, R. J.: Comparison of Two-Dimensional and Three-Dimensional Analyses of Earth-Moon Flight. Proc. Am. Astronautical Soc., Western Regional Meeting (Palo Alto, Calif.), Aug. 1958, pp. 31-1 - 31-15.

7. Mickelwait, A. B., and Booton, R. C., Jr.: Analytical and Numerical Studies of Three-Dimensional Trajectories to the Moon. Paper No. 59-90, Inst. Aero. Sci., June 1959.

8. Anon.: The American Ephemeris and Nautical Almanac for the Year 1959. Nautical Almanac Office, U.S. Naval Observatory, 1957.

9. Jacchia, Luigi G.: The Earth's Gravitational Potential as Derived From Satellites 1957 Beta One and 1958 Beta Two. Special Rep. No. 19, Astrophysical Observatory, Smithsonian Institution, Dec. 6, 1958.

10. Moulton, Forest Ray: An Introduction to Celestial Mechanics. Second rev. ed., The MacMillan Co., c.1914, p. 338.

TABLE I

INJECTION CONDITIONS FOR BASIC CASES

[Injection latitude, 28.46° N.; injection altitude, 300 miles (4,259 miles from center of earth); parabolic velocity at injection, 24.125.8 mph]

Case number	$\frac{V}{V_p}$	γ, deg	ψ, deg	Heading angle, north of east, deg	ρ, deg	a, miles	$\frac{du_m}{dt}$, $\frac{deg}{day}$	Declination of moon at contact, deg
1	0.992	24.00	81.27	0	16.0	238,857	13.195	16.6 S.
2	.996	21.02	95.20	0	16.0	238,857	13.195	16.6 S.
3	1.006	17.26	104.56	0	16.0	238,857	13.195	16.6 S.
4	.992	23.53	83.55	0	16.0	229,100	14.046	16.6 S.
5	.996	20.80	95.47	0	16.0	229,100	14.046	16.6 S.
6	1.006	17.12	104.40	0	16.0	229,100	14.046	16.6 S.
7	.992	24.00	81.76	23.92	18.4	238,857	13.195	0
8	.996	21.02	95.66	23.92	18.4	238,857	13.195	0
9	1.006	17.26	104.95	23.92	18.4	238,857	13.195	0

TABLE II

MISS DISTANCES ON LUNAR SURFACE

$\frac{V}{V_p}$	Distance measured on lunar surface from basic to perturbed impact point, miles						
	Eccentricity of moon's orbit				Earth's oblateness		Solar pertubation
	a = 229,100 miles		a = 238,857 miles		Declination of moon at contact		
	Traj. A	Traj. B	Traj. A	Traj. B	16.6° S.	0°	
0.992	736	147	1,180	230	455	490	280
.996	1,200	44	1,830	147	175	226	120
1.006	1,160	12	995	49	143	159	60

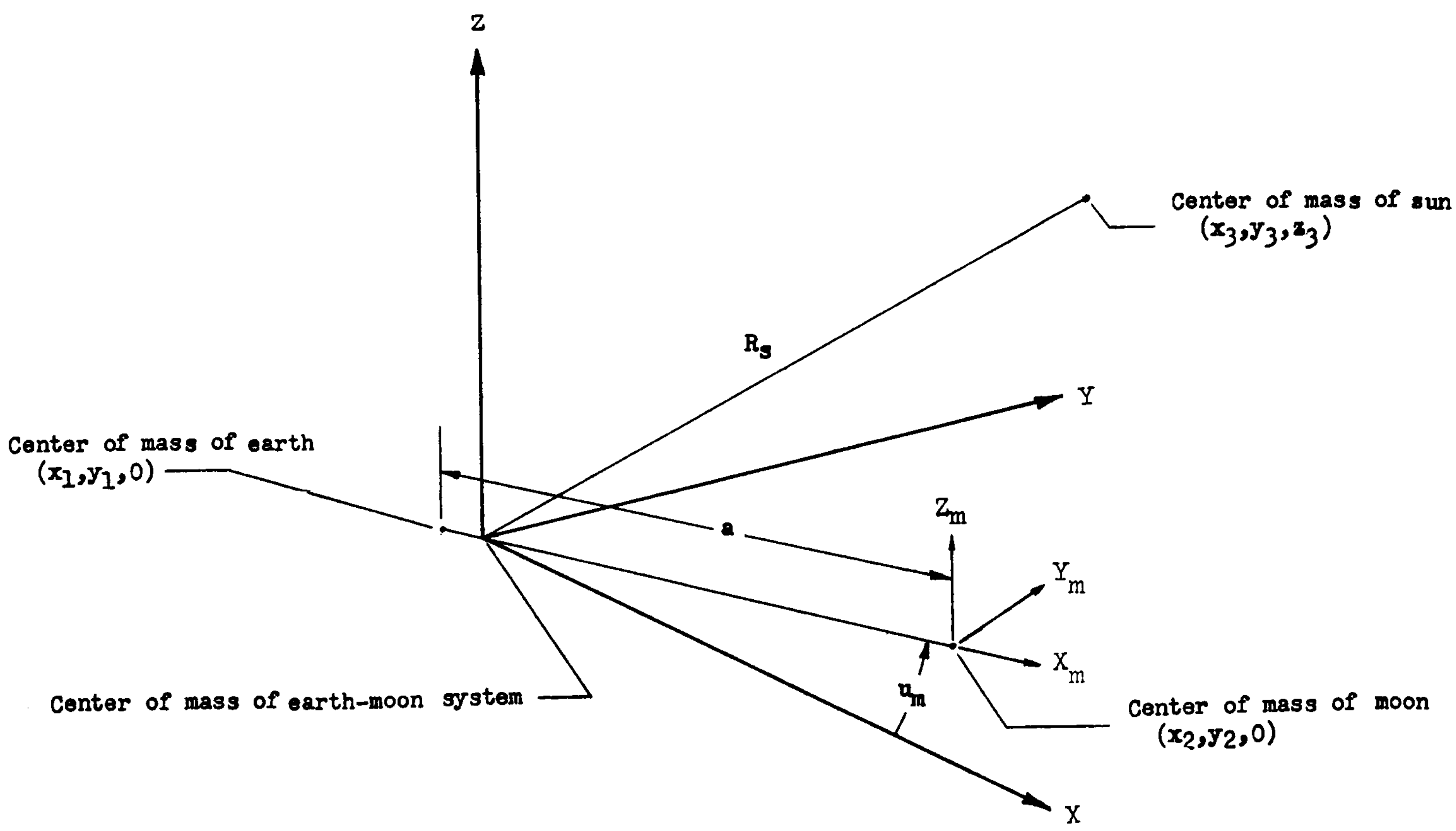


Figure 1.- Coordinate system.

L-800

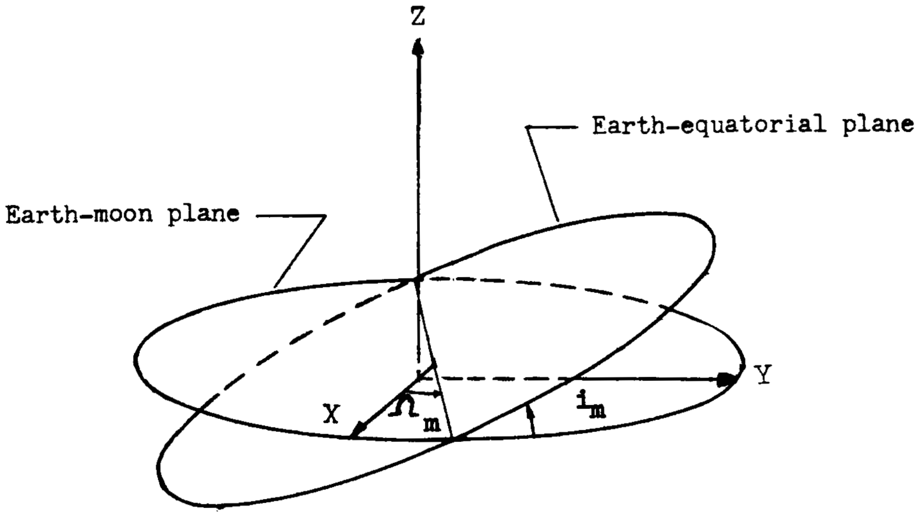


Figure 2.- Illustration of angular relations between coordinate system and earth-equatorial plane.

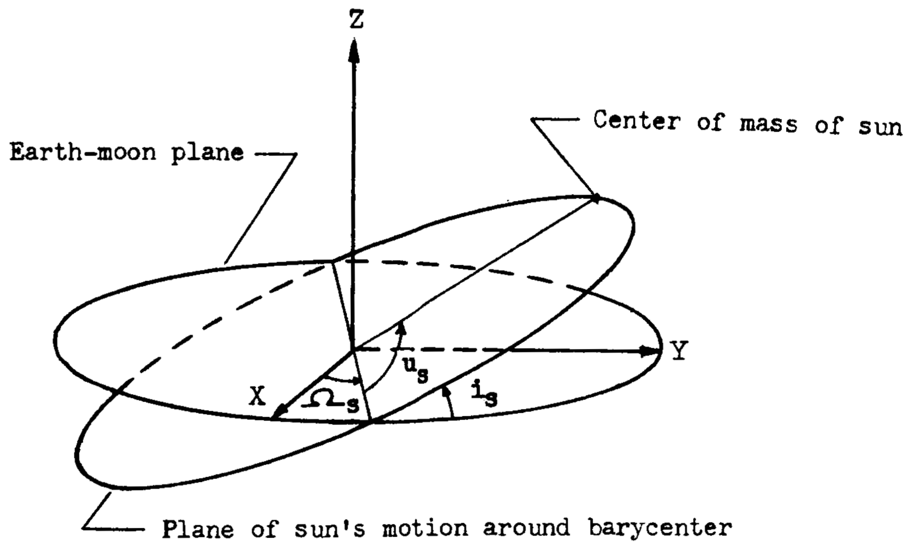


Figure 3.- Illustration of angular relations between coordinate system and sun's plane.

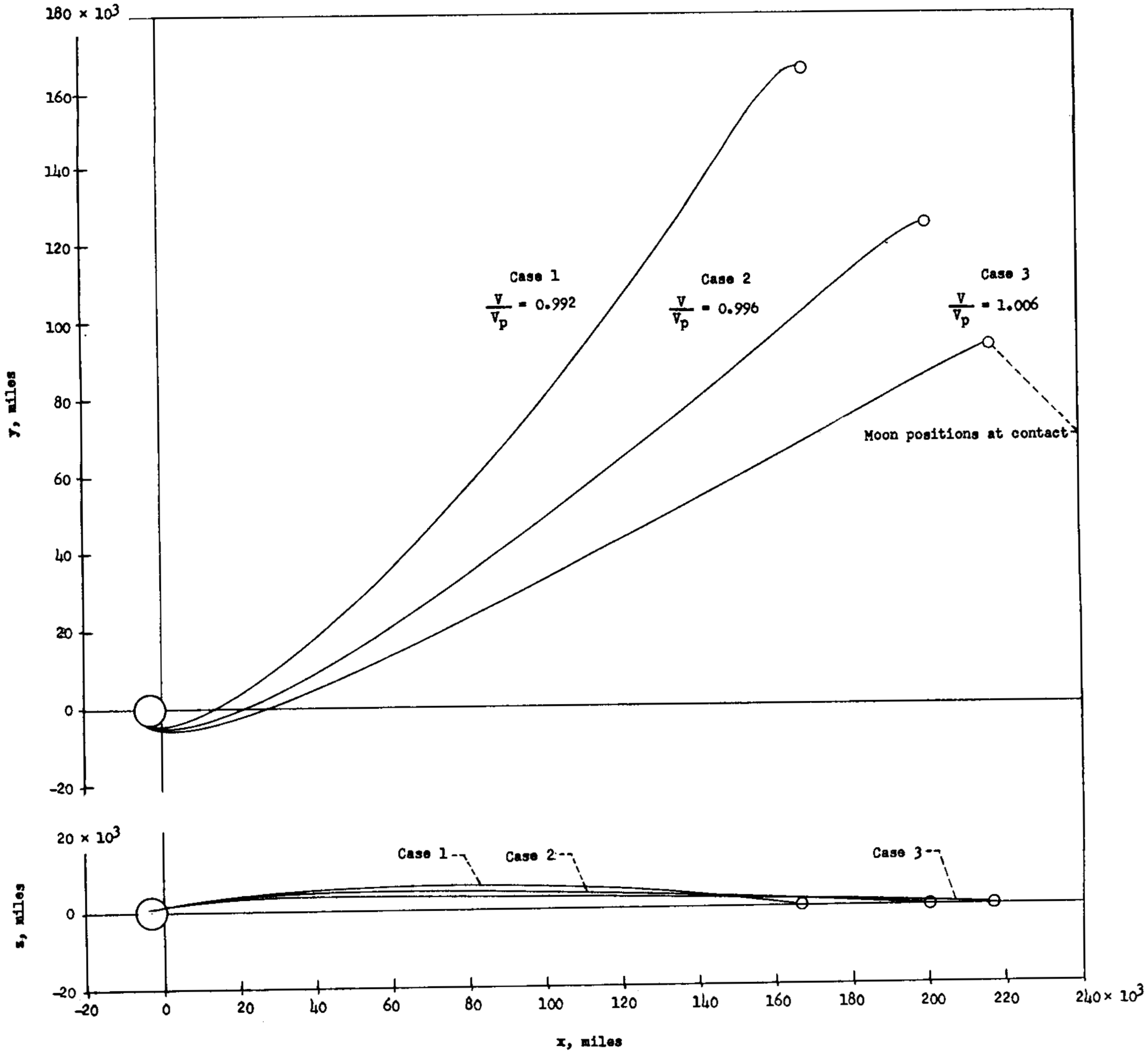


Figure 4.- Typical lunar trajectories in inertial coordinate system. Basic cases 1, 2, and 3.

4D

L-800

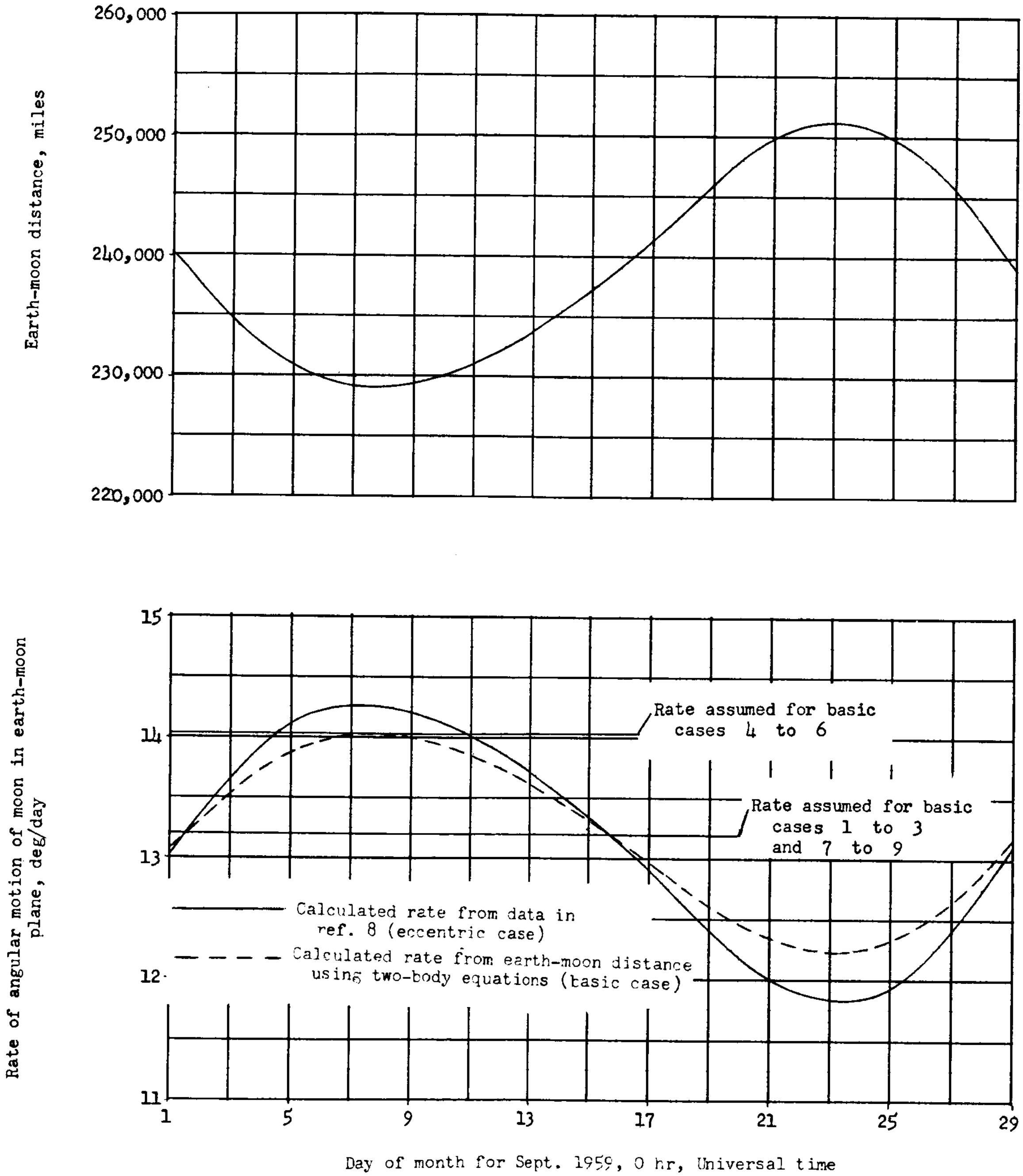
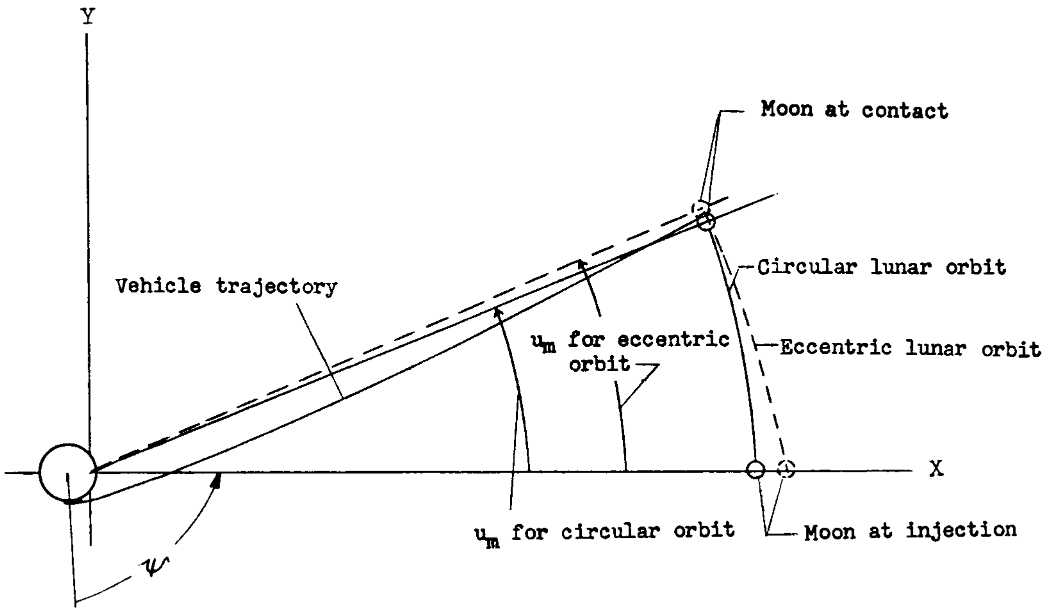
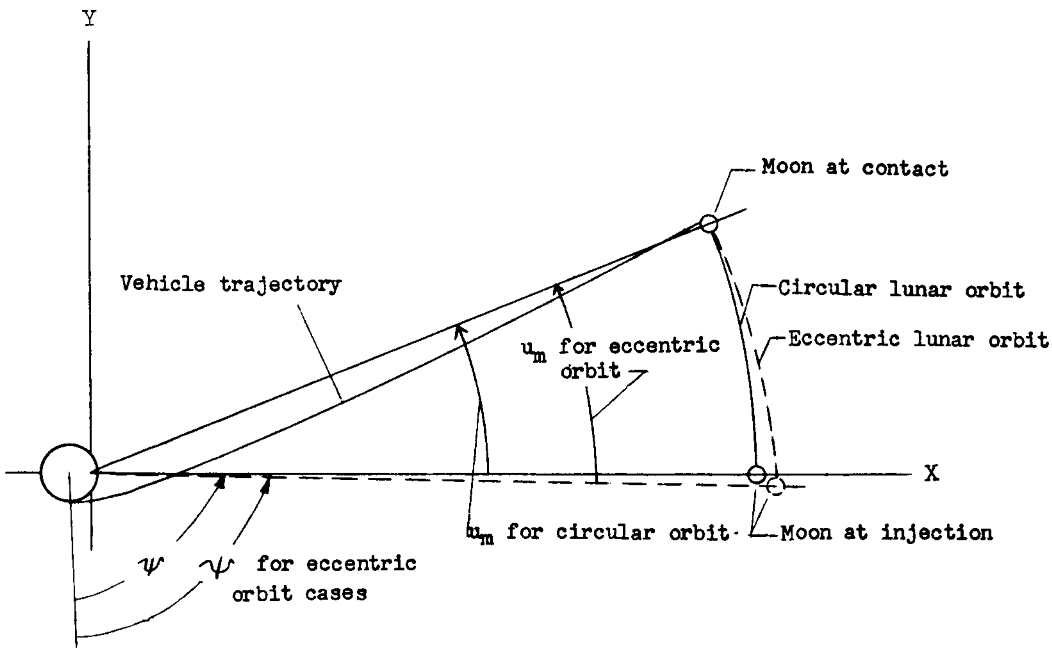


Figure 5.- Typical values of earth-moon distance and rate of angular motion of the moon.



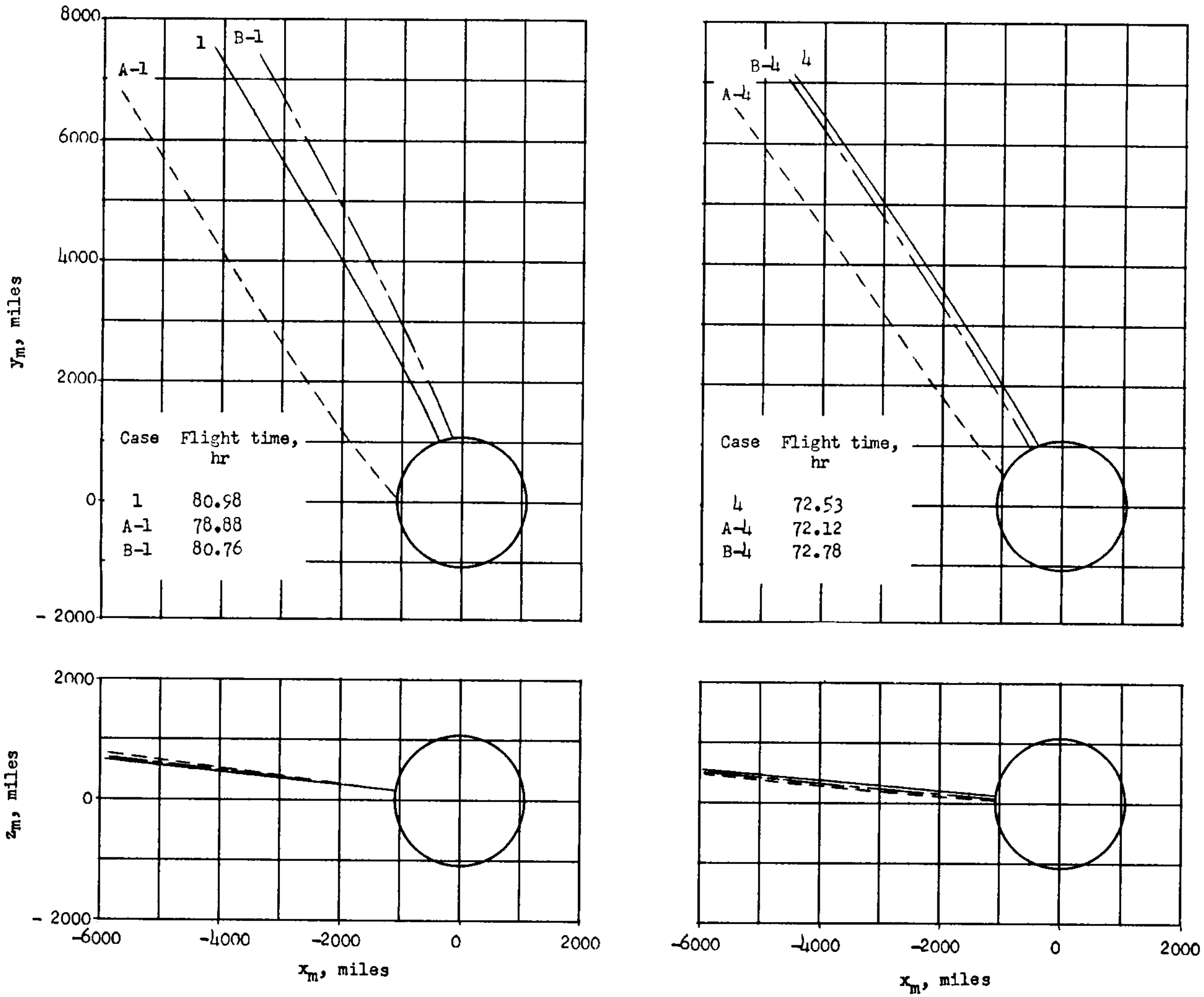
(a) Injection situation for trajectories A.



(b) Injection situation for trajectories B.

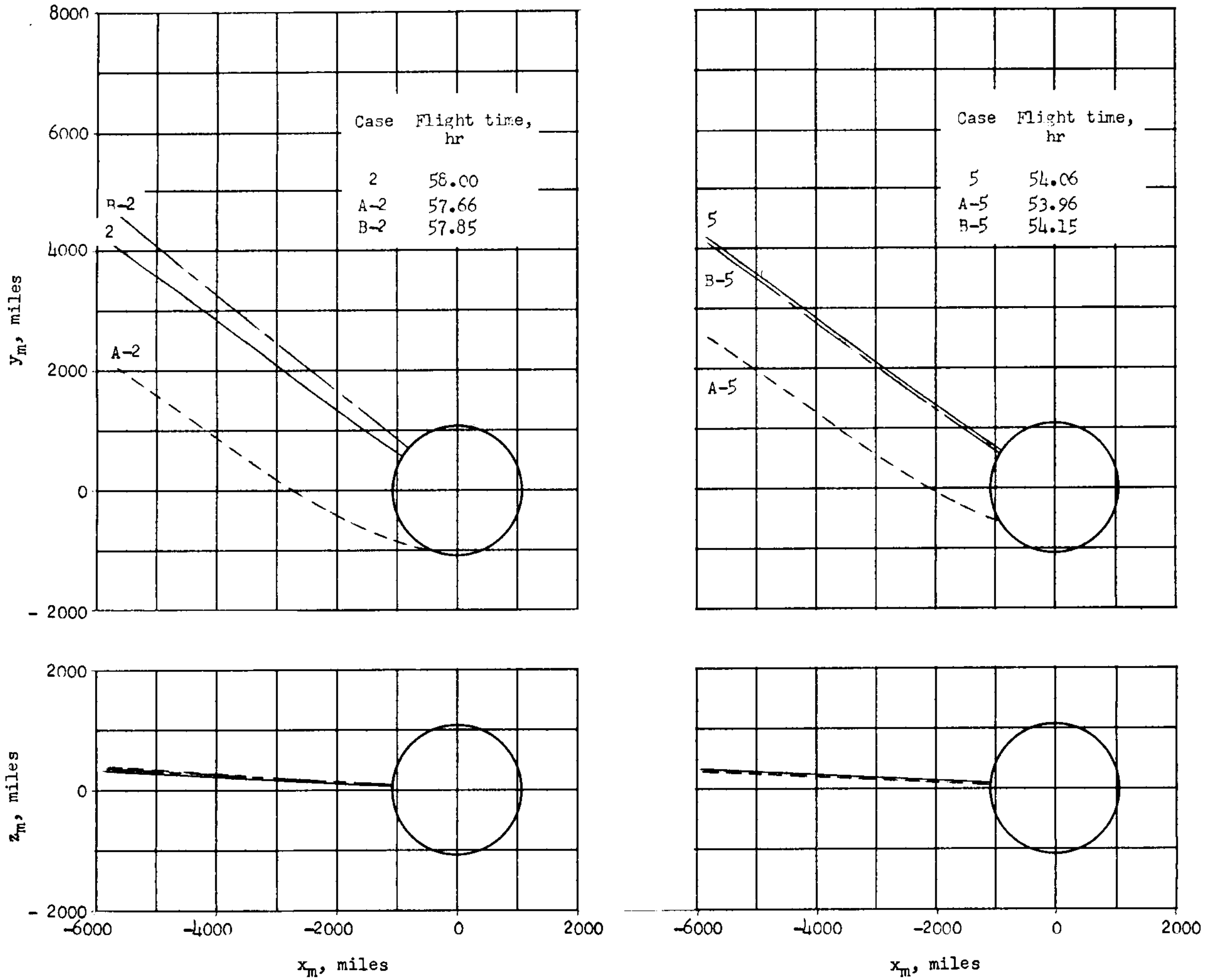
Figure 6.- Schematic illustration of injection situations considered.

L-800



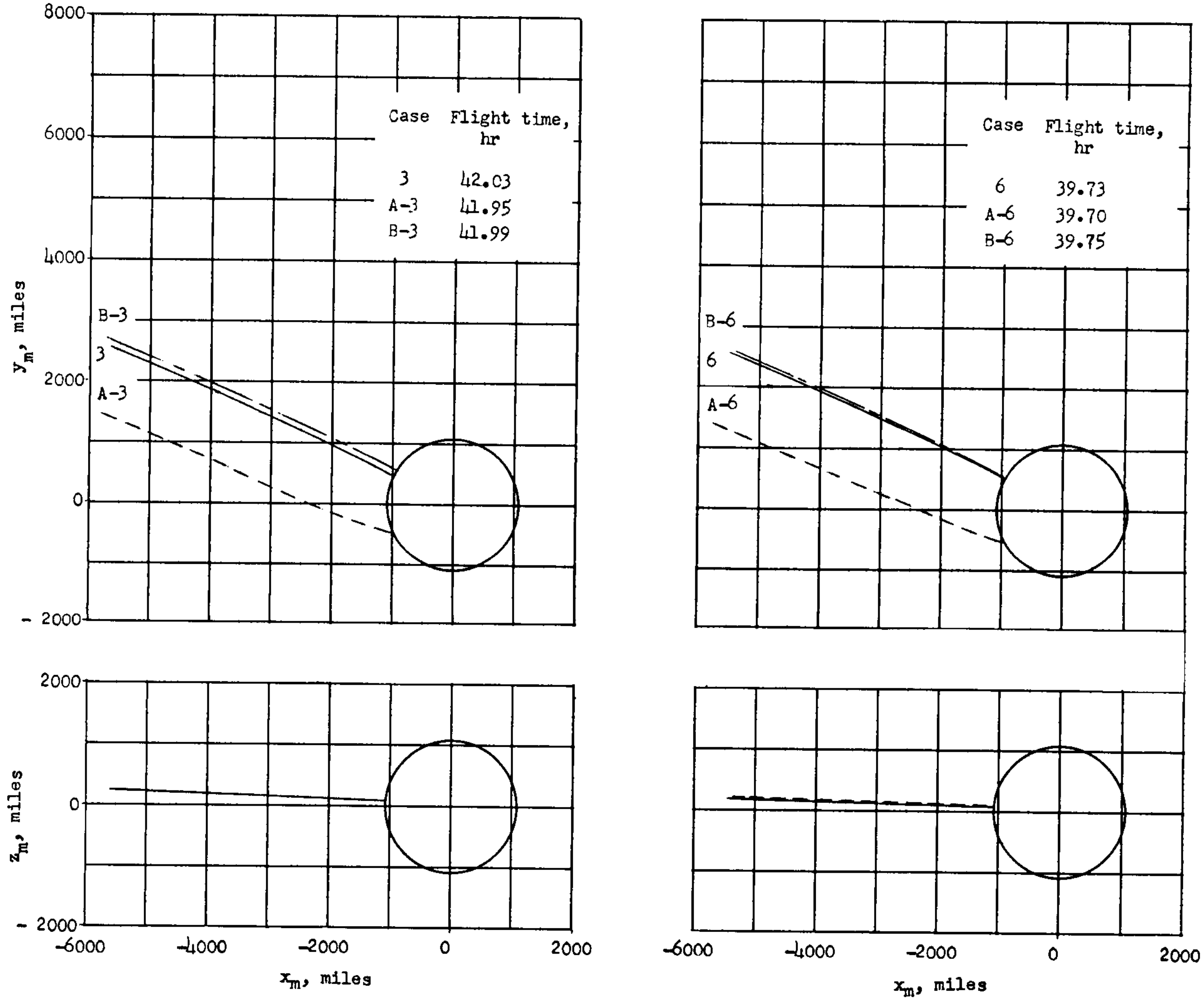
(a) Earth-moon distance, 238,857 miles. (b) Earth-moon distance, 229,100 miles.

Figure 7.- Trajectory comparisons for circular and eccentric lunar orbits. $\frac{V}{V_p} = 0.992$.



(a) Earth-moon distance, 238,857 miles. (b) Earth-moon distance, 229,100 miles.

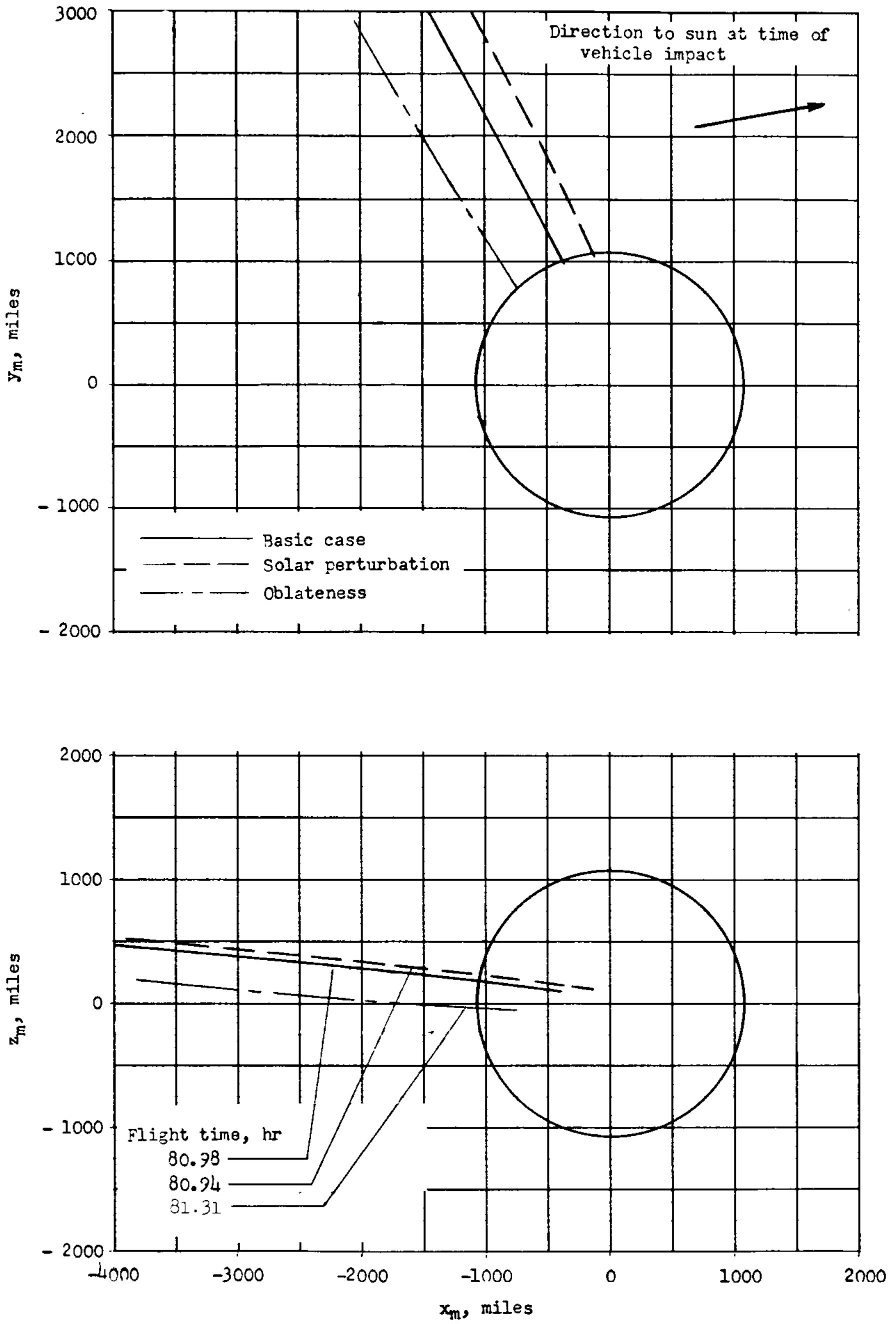
Figure 8.- Trajectory comparisons for circular and eccentric lunar orbits. $\frac{V}{V_p} = 0.996$.



(a) Earth-moon distance, 238,857 miles.

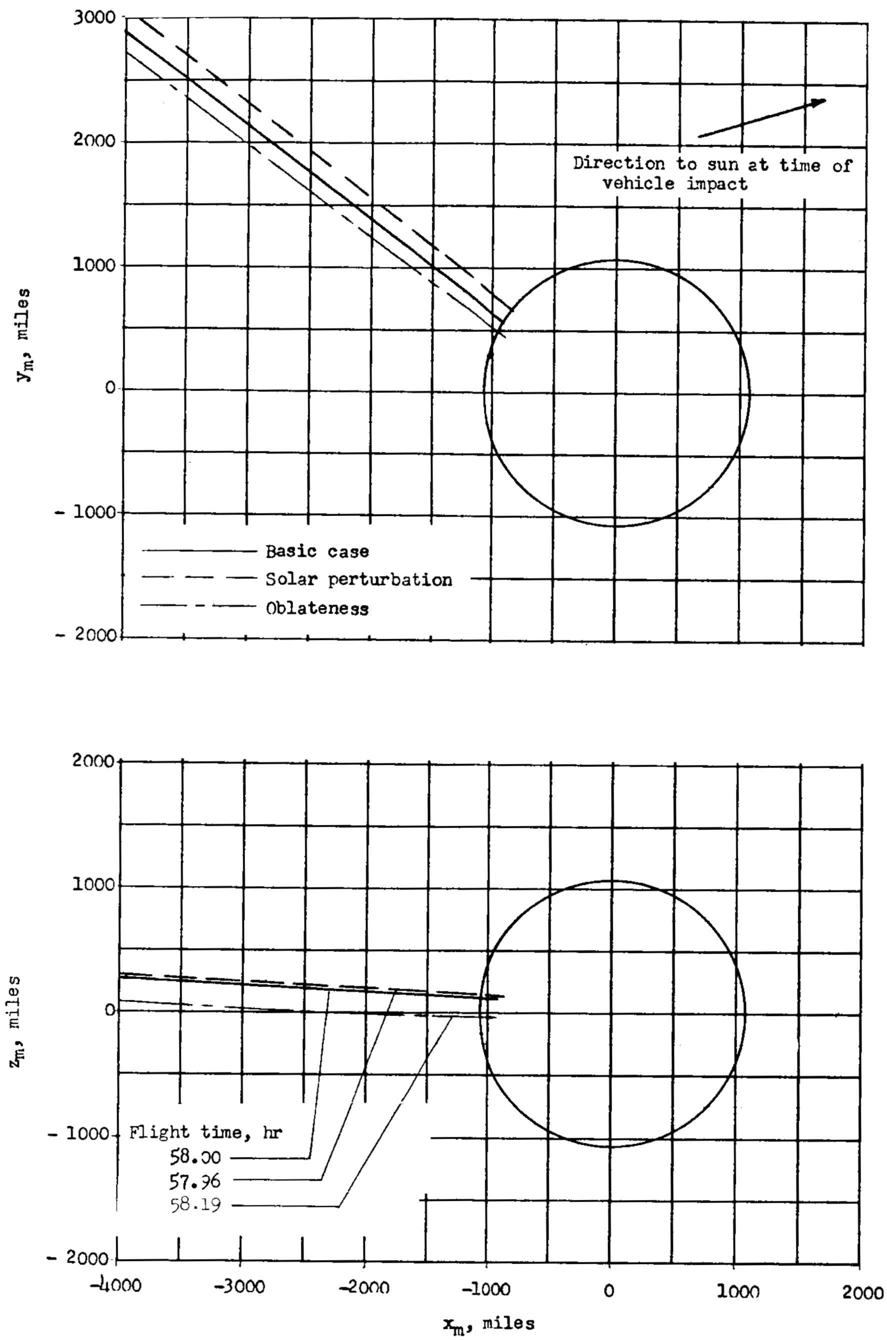
(b) Earth-moon distance, 229,100 miles.

Figure 9.- Trajectory comparisons for circular and eccentric lunar orbits. $\frac{V}{V_p} = 1.006$.



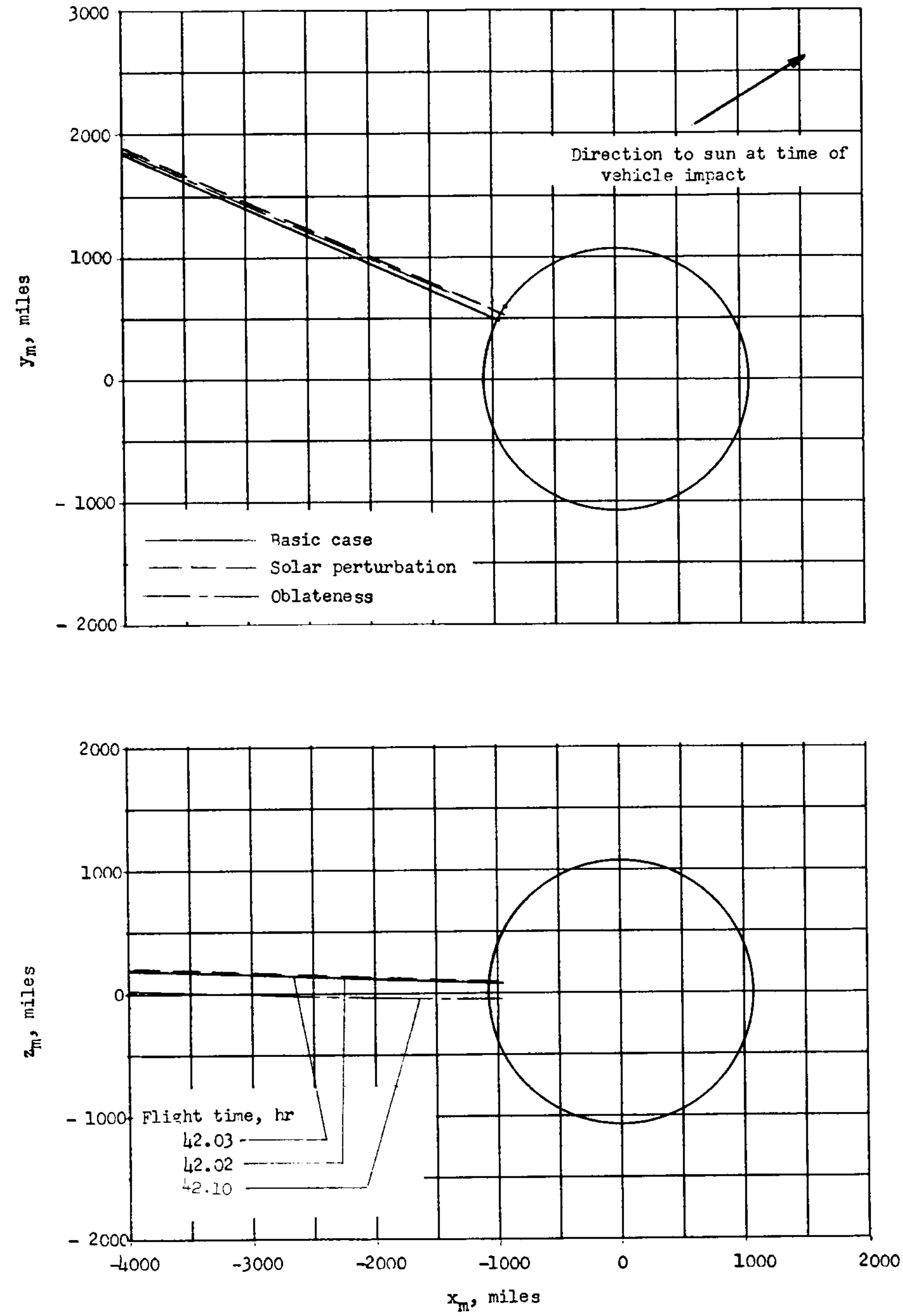
(a) $\frac{V}{V_p} = 0.992.$

Figure 10.- Basic and perturbed trajectories in the region of the moon for due-east injection.



(b) $\frac{V}{V_p} = 0.996.$

Figure 10.- Continued.



(c) $\frac{V}{V_p} = 1.006.$

Figure 10.- Concluded.

L-800

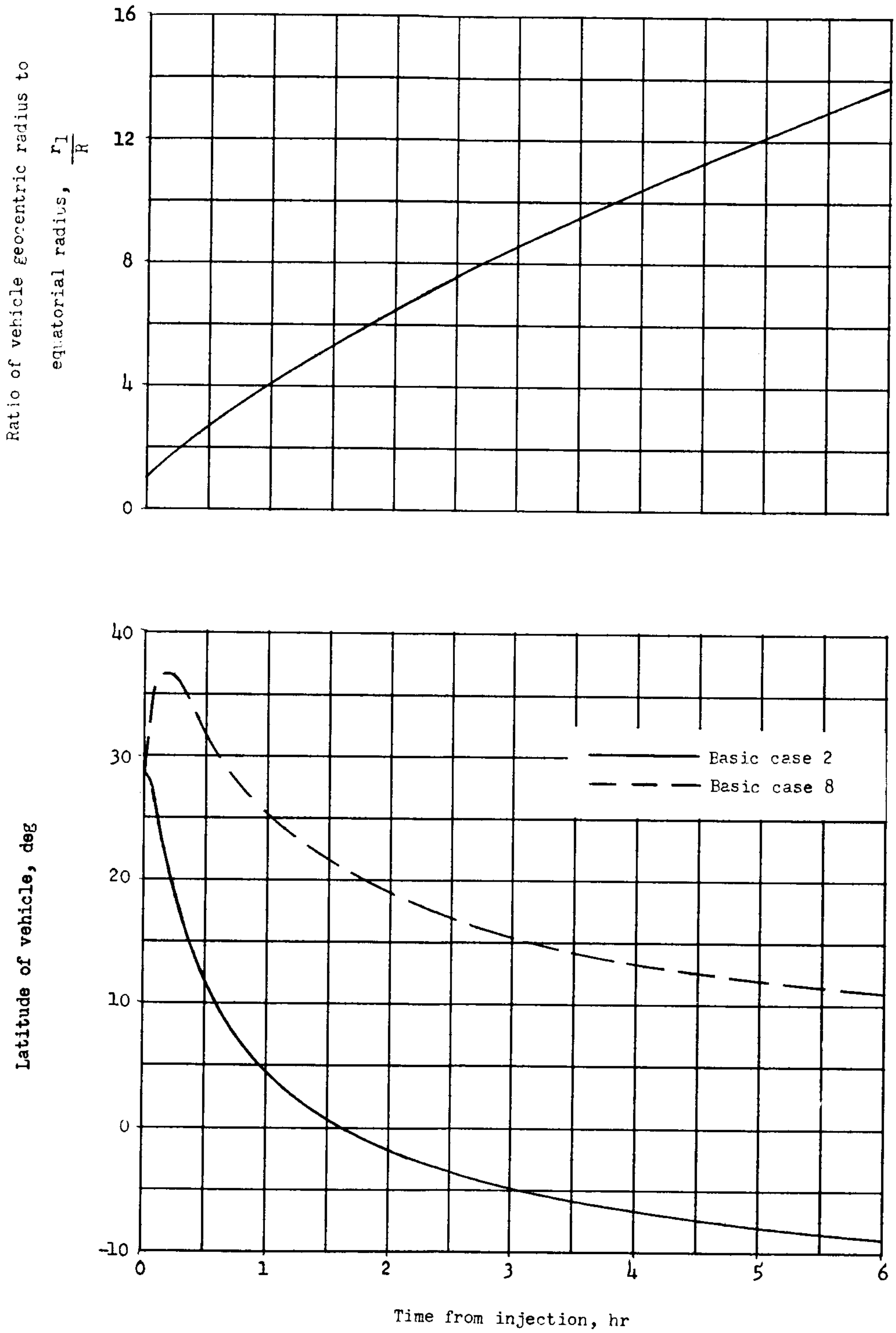
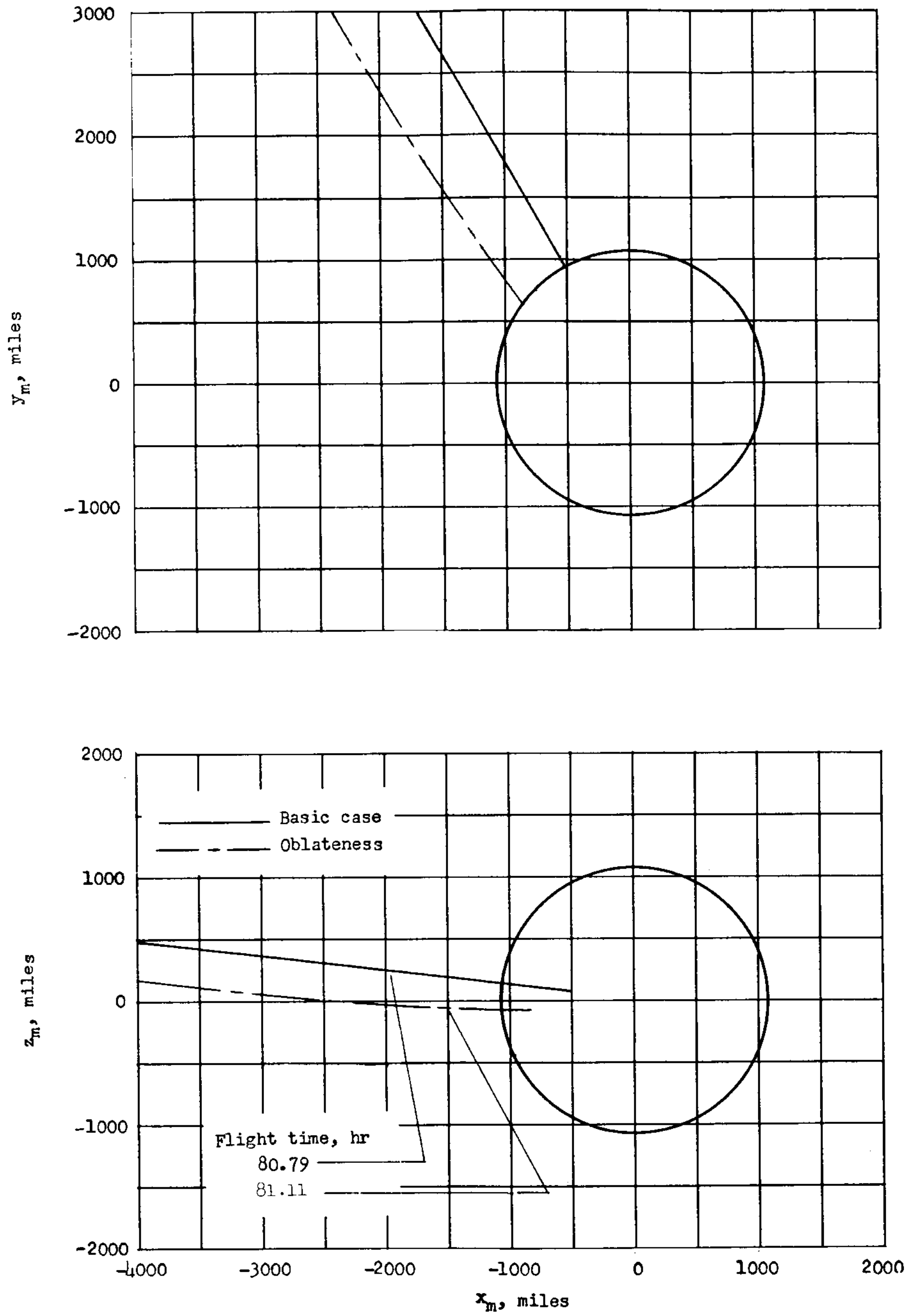


Figure 11.- Latitude and radius time histories for $\frac{V}{V_p} = 0.996$.

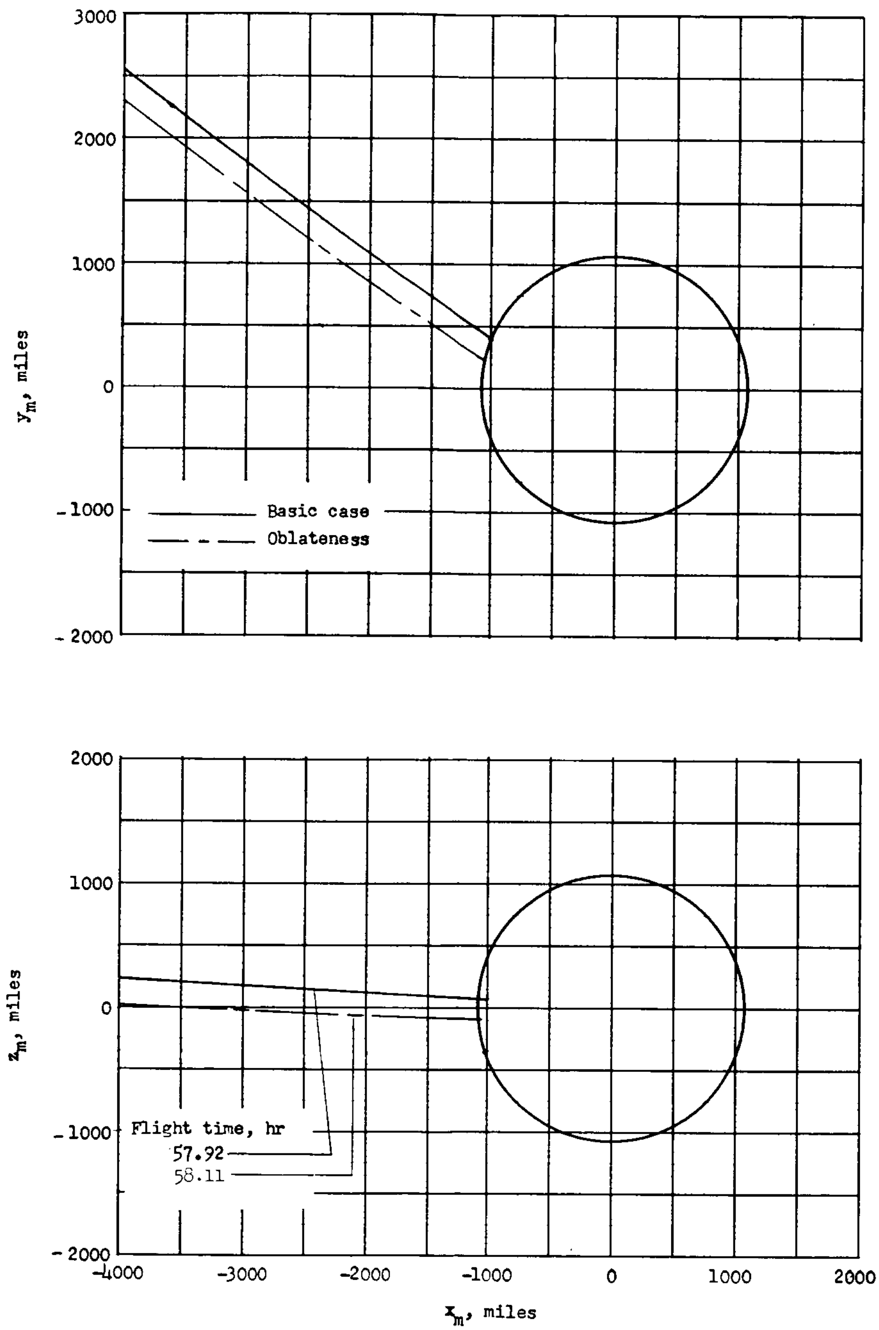


(a) $\frac{V}{V_p} = 0.992.$

Figure 12.- Basic and perturbed trajectories in the region of the moon for north of east injection.

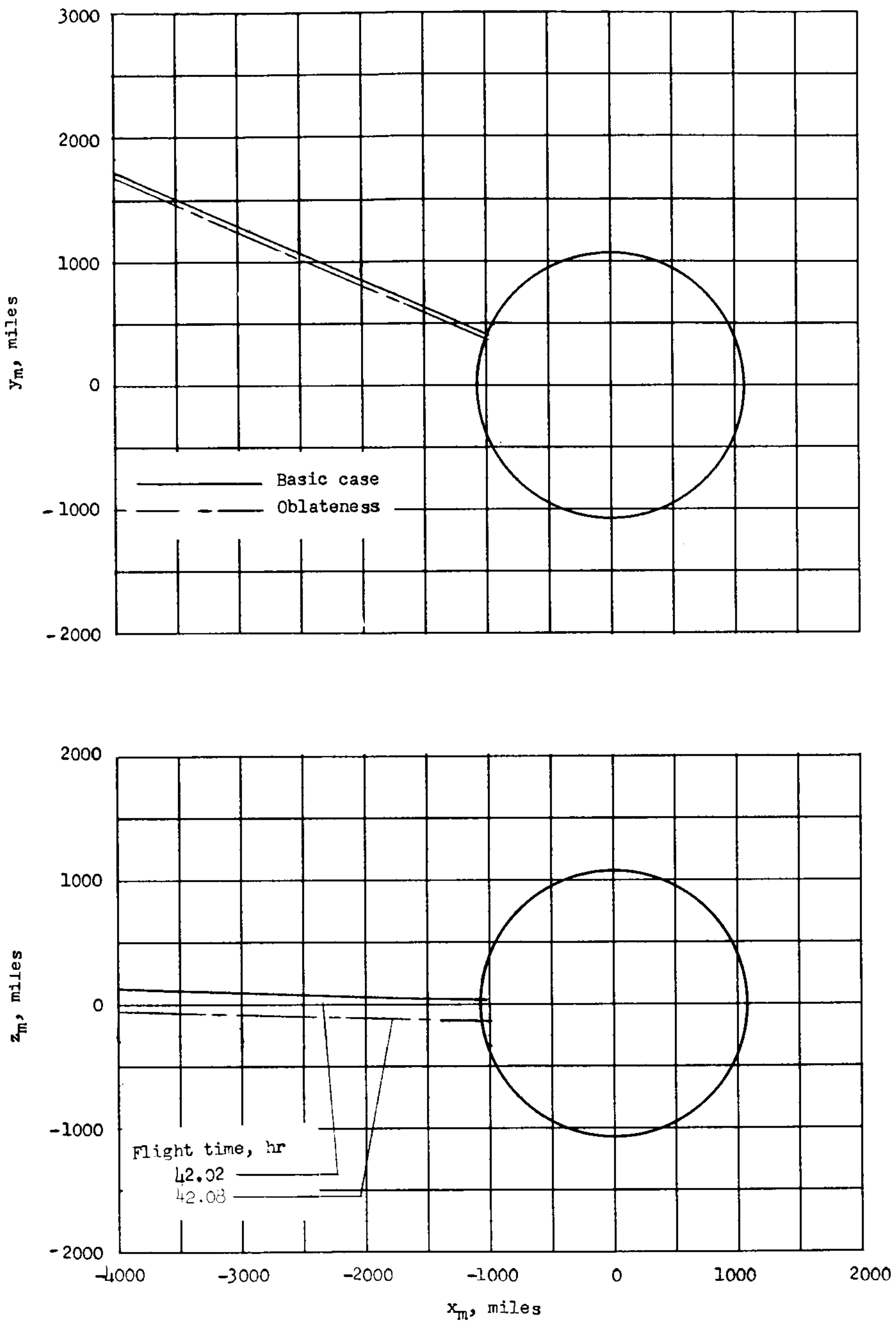
L-800

L-800



(b) $\frac{V}{V_p} = 0.996.$

Figure 12.- Continued.

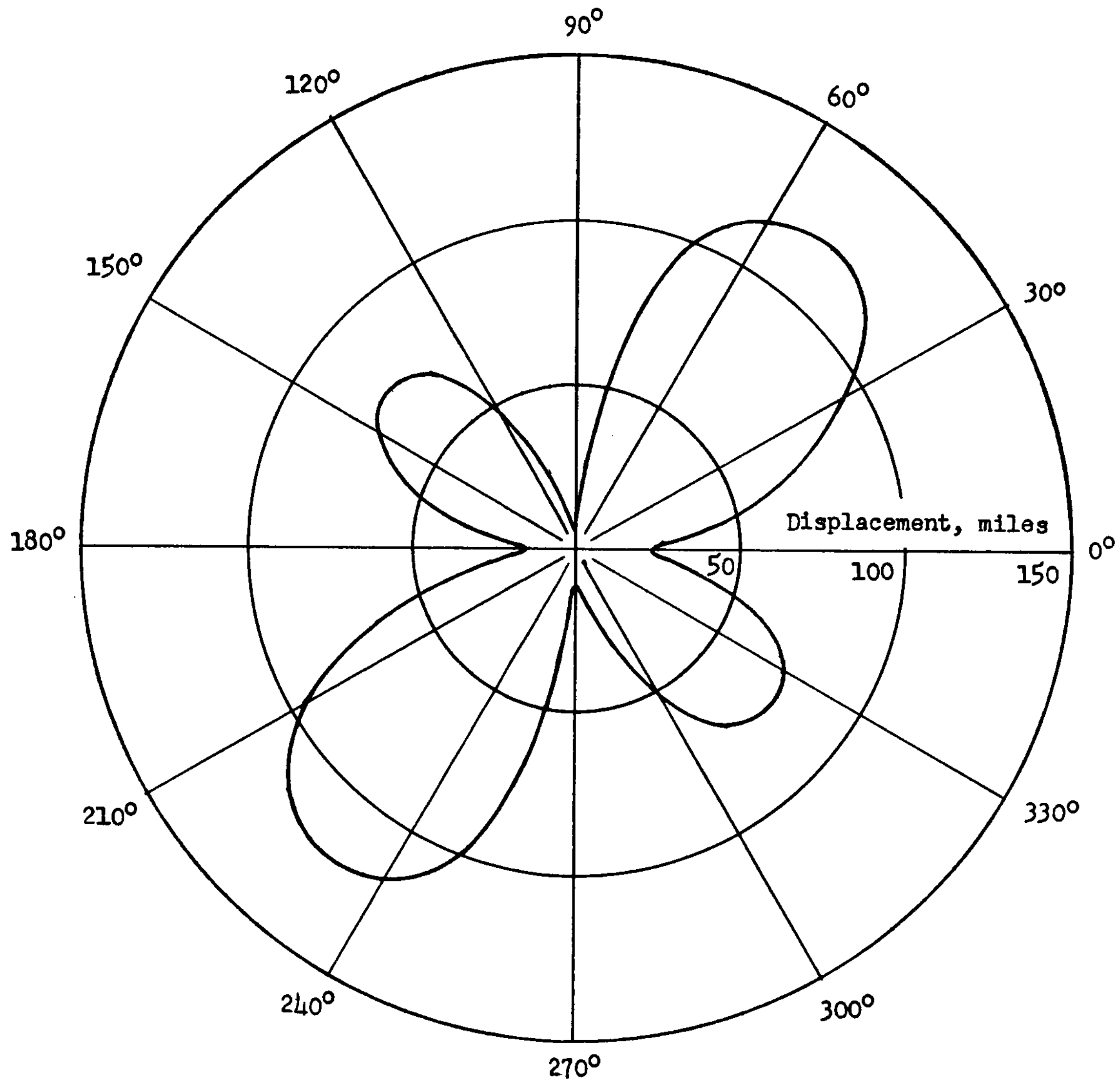


(c) $\frac{V}{V_p} = 1.006$.

Figure 12.- Concluded.

L-800

L-800



Angular position of sun at injection time measured eastward from the X-axis, deg

Figure 13.- Variation of impact point displacement with angular position of sun. $\frac{V}{V_p} = 0.996$.